JOURNAL OF MATERIALS SCIENCE 28 (1993) 1973–1977

Bounds for the elastic constants of a unidirectional fibre composite: a new approach

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Series–parallel and parallel–series models are used to calculate upper and lower bounds for a unidirectional fibre composite. It is shown that this simple approach produces comparatively close bounds which agree well with those calculated from more sophisticated analytical methods. This approach may be used for initial estimates of composite properties, e.g. when characterizing a new material. In some cases this may be sufficient for the design of simple composite structures.

1. Introduction

Following the classic papers of Voigt [1] and Reuss [2] there has been considerable interest in the calculation of upper and lower bounds for the elastic constants of crystalline aggregates, partially oriented polymers [3] and fibre-reinforced composites [4, 5]. In this paper we present a simple method for obtaining bounds for the elastic constants of a unidirectional fibre composite. In one sense, such bounding calculations have been superseded in recent years by finite-element methods [6] and even more recently by analytical methods [7] which give exact solutions. We will, however, show that our approach gives surprisingly close upper and lower bounds for all the elastic constants, and therefore because of its simplicity may be preferable for some purposes to the much more lengthy exact procedures.

2. Theory

2.1. The model composite

A model composite is assumed, consisting of uniaxially oriented square cross-section fibres equally spaced in a homogeneous matrix (Fig. 1). For the purposes of the present simple calculation, it is only necessary to consider an elementary cell of square cross-section with a fibre placed in one corner (Fig. 2).

2.2. Assumption

1. There is a perfect bond between the matrix and the fibres.
2. Each part of the elementary unit cell (fibre or matrix) behaves as a homogeneous elastic material.
3. There is continuity of strain between the two phases.
4. Local torques produced by stresses within the two phases are ignored.

2.3. Definition of elastic constants

Although this approach is applicable for the case where the matrix shows transverse isotropy with the axis of symmetry parallel to the fibre axis, it will be assumed for the sake of simplifying the algebra that the matrix is isotropic. The mechanical properties of the matrix and the fibres can then be described most simply by their engineering elastic constants. These are E_m (tensile modulus) and ν_m (Poisson's ratio) for the matrix and five independent constants $E_{33}, E_{11}, \nu_{13}, \nu_{12}$ and G_{13} where G_{13} is the longitudinal shear modulus for the fibre. It is also convenient to introduce rigidity ratios

$$\zeta_1 = E_{33}/E_m$$
$$\zeta_2 = E_{11}/E_m \quad (1)$$

which will be used throughout the calculations. An appropriate volumetric reinforcement factor F is used

$$F = \frac{V_f}{V_f + V_m} \quad (2)$$

where V_f, V_m are the fibre and matrix volume fractions, respectively. To obtain geometric relations from the global reinforcement factor F a geometric reinforcement factor f is then introduced, such that $f = F$. For a square-section fibre, its linear dimensions would be $f^{1/2} \times f^{1/2}$, so that remaining matrix has dimensions $1 \times (1 - f^{1/2})$ and $f^{1/2} \times (1 - f^{1/2})$. This geometric factor may be further modified, depending on the structure of the composite.

As a quadrilateral structure of cross-section of the composite is uncommon, an "equivalent reinforcement factor" f can be introduced depending on the actual structure of the composite cross-section. For the most probable hexagonal structure, from the ratio of free volumes (for cylindrical fibres) it follows that

$$f = \frac{2}{3^{1/2}} F \quad (3)$$

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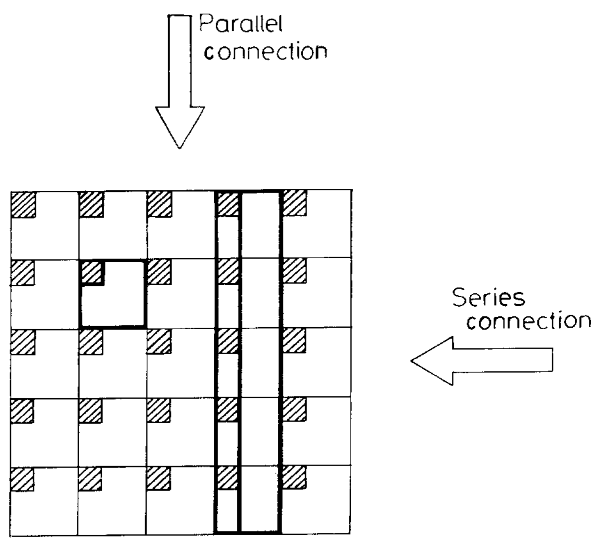

Figure 1 The model composite.

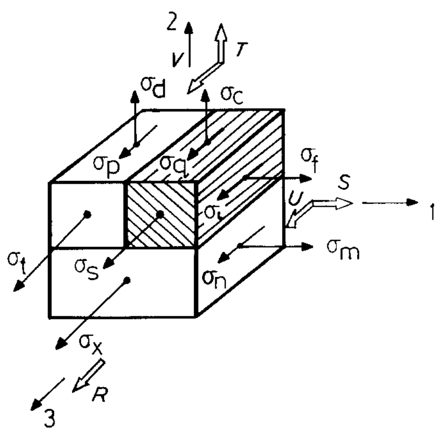

Figure 2 Elementary cell in the model composite. *R, S, T, U* and *V* are average external stresses.

If full compatibility of internal strains is to be achieved as well, a further correction to the reinforcement factor can be made, e.g. as presented in a paper by one of the authors [7], leading to a value

$$f = \frac{2(3^{1/2})}{\pi} F \tag{3a}$$

which assumes that the unit material cell is embedded in the composite material, rather than in the polymeric material.

2.4. Calculation of bounds

The unidirectional fibre composite has five independent elastic constants E_{33}, E_{11}, ν_{13}, ν_{12} and G_{13} for which lower and upper bounds will be calculated as E^{L}_{33}, E^{L}_{11}, ν^{L}_{13}, ν^{L}_{12}, G^{L}_{13} and E^{U}_{33}, E^{U}_{11}, ν^{U}_{13}, ν^{U}_{12}, G^{U}_{13}, respectively. The lower and upper bounds are obtained by considering the two possible series–parallel connectivity situations as illustrated in Fig. 2. In two dimensions these schemes are familiar as the Takayanagi series–parallel and parallel–series models (see, for example, Ward [8] p. 300).

For this case two separate sets of boundary conditions can be considered for simplification of calculations, based on the independence, in principal coordinates of orthotropic materials, of shear and volumetric strains. Assuming external shear stresses to be zero ($U = V = 0$) the following relations are valid.

Static equilibrium conditions of the unit volume are represented by

$$\begin{aligned} \sigma_r(1 - f^{1/2}) + \sigma_s f + \sigma_t f^{1/2}(1 - f^{1/2}) &= R \\ \sigma_f f^{1/2} + \sigma_m(1 - f^{1/2}) &= S \\ \sigma_c f^{1/2} + \sigma_d(1 - f^{1/2}) &= T \end{aligned} \tag{4}$$

Strains in the layer containing the reinforcement denoted below by primes (′) are of the form

$$\begin{aligned} \varepsilon'_{11} &= \left[\frac{\sigma_f}{E_{11}} - \frac{\sigma_c}{E_{11}}\nu_{21} - \frac{\sigma_s}{E_{11}}\nu_{31}\right] f^{1/2} \\ &\quad + \left[\frac{\sigma_f}{E_m} - \frac{\sigma_d}{E_m}\nu_m - \frac{\sigma_t}{E_m}\nu_m\right](1 - f^{1/2}) \\ \varepsilon'_{22} &= \frac{\sigma_c}{E_{11}} - \frac{\sigma_f}{E_{11}}\nu_{21} - \frac{\sigma_s}{E_{11}}\nu_{31} \\ &= \frac{\sigma_d}{E_m} - \frac{\sigma_f}{E_m}\nu_m - \frac{\sigma_t}{E_m}\nu_m \\ \varepsilon'_{33} &= \frac{\sigma_s}{E_{33}} - \frac{\sigma_c}{E_{33}}\nu_{13} - \frac{\sigma_f}{E_{33}}\nu_{13} \\ &= \frac{\sigma_t}{E_m} - \frac{\sigma_d}{E_m}\nu_m - \frac{\sigma_f}{E_m}\nu_m \end{aligned} \tag{5}$$

where compatibility of strains between the two components of the layer are also ensured. Strains in the remaining, unreinforced part of the composite are given by

$$\begin{aligned} \varepsilon_{11} &= \frac{\sigma_m}{E_m} - \frac{\sigma_r}{E_m}\nu_m - \frac{T}{E_m}\nu_m \\ \varepsilon_{22} &= \frac{T}{E_m} - \frac{\sigma_m}{E_m}\nu_m - \frac{\sigma_r}{E_m}\nu_m \\ \varepsilon_{33} &= \frac{\sigma_r}{E_m} - \frac{\sigma_m}{E_m}\nu_m - \frac{T}{E_m}\nu_m \end{aligned} \tag{6}$$

The only remaining conditions to be fulfilled are the compatibility conditions between the reinforced and unreinforced layers of the composite. These are

$$\begin{aligned} \varepsilon_{11} &= \varepsilon'_{11} = \bar{\varepsilon}_{11} \\ \varepsilon'_{22} f^{1/2} + \varepsilon_{22}(1 - f^{1/2}) &= \bar{\varepsilon}_{22} \\ \varepsilon_{33} &= \varepsilon'_{33} = \bar{\varepsilon}_{33} \end{aligned} \tag{7}$$

where the bar ($^-$) denotes average values of strain for a composite as a whole.

Using Equations 4–7 it is possible, after inserting Equations 1 and 2, to obtain a set of equations

$$\begin{aligned} \sigma_r(1 - f^{1/2}) + \sigma_s f + \sigma_t f^{1/2}(1 - f^{1/2}) &= R \\ \sigma_f f^{1/2} + \sigma_m(1 - f^{1/2}) &= S \\ \sigma_c f^{1/2} + \sigma_d(1 - f^{1/2}) &= T \\ \sigma_s - \sigma_c\nu_{21} - \sigma_f\nu_{21} - \zeta_1(\sigma_t - \sigma_d\nu_m - \sigma_f\nu_m) &= 0 \end{aligned} \tag{8}$$

$$\sigma_c - \sigma_f v_{32} - \sigma_s v_{12} - \zeta_2(\sigma_d - \sigma_f v_m - \sigma_t v_m) = 0$$

$$\sigma_t - \sigma_d v_m - \sigma_f v_m - \sigma_r + \sigma_m v_m = -T v_m$$

$$(\sigma_f - \sigma_c v_{32} - \sigma_s v_{12}) f^{1/2} + \zeta_2(\sigma_f - \sigma_d v_m - \sigma_t v_m)$$

$$\times (1 - f^{1/2}) - \zeta_2(\sigma_m - \sigma_r v_m) = -\zeta_2 T v_m$$

describing the case under consideration.

Assuming now homogeneity of the composite, loaded as depicted in Fig. 3, constitutive relations for the unit cell take the form

$$\bar{\varepsilon}_{11} = \frac{1}{E_{11}}(S - \bar{v}_{21} T - \bar{v}_{31} R)$$

$$\bar{\varepsilon}_{22} = \frac{1}{E_{11}}(T - \bar{v}_{31} R - \bar{v}_{21} S) \qquad (9)$$

$$\bar{\varepsilon}_{33} = \frac{1}{E_{33}}(R - \bar{v}_{13} S - \bar{v}_{13} T)$$

where the bar ($^-$) describes average values for the composite as a whole. Combining Equations 7 and 9, values of bounds for particular constants can be found.

It can be also seen both from Equations 9 and Fig. 2 that interchanging external stresses S and T leads to serial and parallel connection of the two layers in respective directions.

There are finally three cases of interest:

$$\text{Case 1} \quad R = 1 \quad S = 0 \quad T = 0$$
$$\text{Case 2} \quad R = 0 \quad S = 1 \quad T = 0 \qquad (10)$$
$$\text{Case 3} \quad R = 0 \quad S = 0 \quad T = 1$$

which will be treated independently.

Case 1

$$\frac{1}{E_{33}} = \frac{1}{E_m}(\sigma_r - \sigma_m v_m) \qquad (11a)$$

$$-\frac{\bar{v}_{31}}{E_{11}} = \frac{1}{E_m}(\sigma_m - \sigma_r v_m) \qquad (11b)$$

$$-\frac{\bar{v}_{31}}{E_{11}} = \frac{f^{1/2}}{E_m}(\sigma_d - \sigma_f v_m - \sigma_t v_m) - \frac{1 - f^{1/2}}{E_m}(\sigma_r + \sigma_m) v_m \qquad (11c)$$

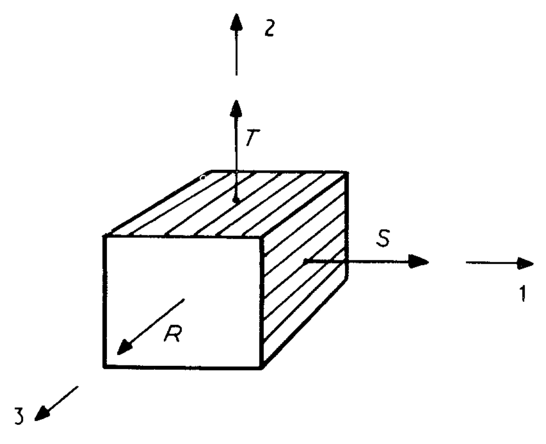

Figure 3 Loads on the composite.

Case 2

$$-\frac{\bar{v}_{13}}{E_{33}} = \frac{1}{E_m}(\sigma_r - \sigma_m v_m) \qquad (12a)$$

$$\frac{1}{E_{11}} = \frac{1}{E_m}(\sigma_m - \sigma_r v_m) \qquad (12b)$$

$$-\frac{\bar{v}_{21}}{E_{11}} = (\sigma_d - \sigma_f v_m - \sigma_t v_m) \times \frac{f^{1/2}}{E_m} - (\sigma_m - \sigma_r) v_m \frac{1 - f^{1/2}}{E_m} \qquad (12c)$$

Case 3

$$-\frac{\bar{v}_{13}}{E_{33}} = \frac{1}{E_m}(\sigma_r - \sigma_m v_m - v_m) \qquad (13a)$$

$$-\frac{\bar{v}_{21}}{E_{11}} = \frac{1}{E_m}(\sigma_m - \sigma_r v_m - v_m) \qquad (13b)$$

$$\frac{1}{E_{11}} = (\sigma_d - \sigma_f v_m - \sigma_t v_m)\frac{f^{1/2}}{E_m} + (1 - \sigma_m v_m - \sigma_r v_m)\frac{1 - f^{1/2}}{E_m} \qquad (13c)$$

where for calculation purposes the external stresses are taken as unity.

It may be now clearly seen that the proposed approach gives directly bounds for the following:

$\frac{v_{31}}{E_{11}}$ from Equations 11b and 11c

$\frac{v_{13}}{E_{33}}$ from Equations 12a and 13a

$\frac{1}{E_{11}}$ from Equations 12b and 13c

$\frac{v_{21}}{E_{11}}$ from Equations 12c and 13b

and a single value for $1/E_{33}$ given by Equation 11a. To obtain bounds for this modulus the following procedure was adopted. From general symmetry relations for an orthotropic material we have

$$v_{13} = v_{31}\frac{E_{33}}{E_{11}} \qquad (14)$$

This means it is possible to use Equations 11b and 11c to obtain bounds for v_{13} and insert their values into Equations 12a and 13a, thus obtaining the necessary bounds for E_{33}.

However, analytical solutions of a set of seven simultaneous Equations 8 for the unknown values of stresses σ_c, σ_d, σ_f, σ_m, σ_r, σ_s, σ_t are totally unmanageable. To obtain the final results a numerical computer program was therefore prepared and further numerical calculations were performed.

The second separate case is when

$$R = S = T = 0 \qquad (15)$$

where either $U = 1$ and $V = 0$ or $U = 0$ and $V = 1$. This case is sufficiently simplified for final analytical values to be obtained for the bounds of the remaining constant $\bar{G}_{13}$, while the constant $\bar{G}_{12}$ can be easily calculated from the relation

$$\bar{G}_{12} = \frac{E_{11}}{2(1 + \nu_{12})} \tag{16}$$

For the case when $U = 1$ and $V = 0$ it is easily found that all stresses except σ_l and σ_n can be assumed to be zero and there is only one equation of equilibrium

$$\sigma_l f^{1/2} + \sigma_n(1 - f^{1/2}) = 1 \tag{17}$$

Strain compatibility conditions demand that

$$\frac{\sigma_l}{G_{13}} f^{1/2} + \frac{\sigma_l}{G_m}(1 - f^{1/2}) = \frac{\sigma_n}{G_m} \tag{18}$$

while the constitutive relation gives

$$\frac{1}{\bar{G}_{13}} = \frac{\sigma_n}{G_m} \tag{19}$$

Performing the necessary calculation, it is easily found that the upper bound of this modulus is

$$G^{U}_{31} = G_m \frac{G_m f^{1/2}(1 - f^{1/2}) + G_{13}[1 - f^{1/2}(1 - f^{1/2})]}{G_m f^{1/2} + G_{13}(1 - f^{1/2})} \tag{20}$$

Similarly a straightforward solution can be found for the case $U = 0$, $V = 1$, when only σ_p and σ_q are not zero. The only equilibrium equation is

$$\sigma_p(1 - f^{1/2}) + \sigma_q f^{1/2} = 1 \tag{21}$$

while the only compatibility condition is

$$\frac{\sigma_q}{G_{31}} = \frac{\sigma_p}{G_m} \tag{22}$$

and the constitutive equation takes the form

$$\frac{1}{\bar{G}_{13}} = \frac{\sigma_p}{G_m} f^{1/2} + \frac{1}{G_m}(1 - f^{1/2}) \tag{23}$$

This leads directly to the value of the lower bound of the modulus, which takes the form

$$G^{L}_{13} = \frac{G_m(1 - f^{1/2}) + G_{13} f^{1/2}}{G_m[1 - f^{1/2}(1 - f^{1/2})] + G_{13} f^{1/2}(1 - f^{1/2})} \tag{24}$$

3. Results for a typical system

The theory outlined has been examined by calculating the values of the elastic constants for a unidirectional glass fibre composite where we have assumed $E_m = 3.0$ GPa, $\nu_m = 0.3$; $E_f = 70$ GPa, $\nu_f = 0.25$. These data were used to produce exemplary graphs. Other values were used as presented in the tables.

The results are shown in Tables I–IV and Fig. 4. It can be seen that the upper and lower bounds are always quite close, comparing very favourably with

TABLE I Model composite calculations for matrix $E_m = 1.0$ GPa and $\nu_m = 0.45$, reinforcement $E_f = 120$ GPa and $\nu_f = 0.3$, volume fraction $f = 0.3$

Parameter	Lower bound	Upper bound
E_{33} (GPa)	38.03	46.90
E_{11} (GPa)	5.44	6.93
ν_{13}	0.37	0.42
ν_{12}	0.35	0.44
G_{13} (GPa)	0.63	0.82
G_{12} (GPa)	2.02	2.41

TABLE II Model composite calculations for matrix $E_m = 4.0$ GPa and $\nu_m = 0.35$, reinforcement $E_f = 80$ GPa and $\nu_f = 0.25$, volume fraction $f = 0.6$

Parameter	Lower bound	Upper bound
E_{33} (GPa)	55.86	57.48
E_{11} (GPa)	22.74	24.84
ν_{31}	0.27	0.28
ν_{12}	0.17	0.19
G_{13} (GPa)	6.23	6.94
G_{12} (GPa)	9.72	10.48

TABLE III Model composite calculations for matrix $E_m = 30$ GPa and $\nu_m = 0.35$, reinforcement $E_f = 220$ GPa and $\nu_f = 0.25$, volume fraction $f = 0.5$

Parameter	Lower bound	Upper bound
E_{33} (GPa)	136.53	143.12
E_{11} (GPa)	90.77	97.40
ν_{13}	0.28	0.29
ν_{12}	0.24	0.26
G_{13} (GPa)	27.79	30.73
G_{12} (GPa)	36.63	38.77

TABLE IV Model composite calculations for matrix $E_m = 4.0$ GPa and $\nu_m = 0.3$, reinforcement $E_f = 1.0$ GPa and $\nu_f = 0.4$, volume fraction $f = 0.5$

Parameter	Lower bound	Upper bound
E_{33} (GPa)	2.23	2.32
E_{11} (GPa)	2.08	2.13
ν_{13}	0.35	0.35
ν_{12}	0.33	0.34
G_{13} (GPa)	0.70	0.75
G_{12} (GPa)	0.78	0.79

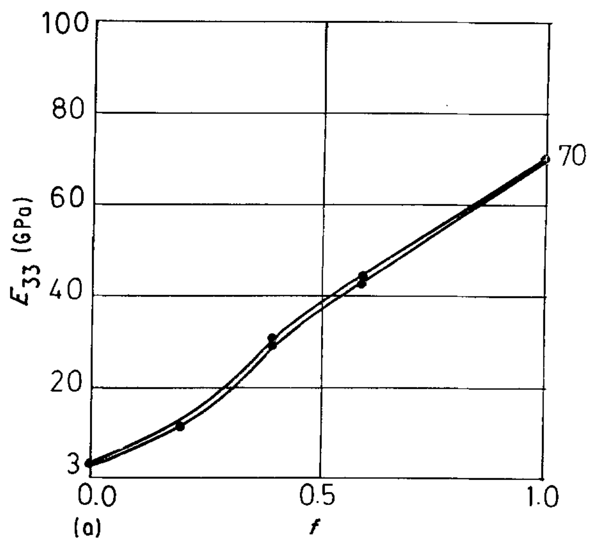

Figure 4 Dependence of bounds on the reinforcement factor f: (a) E_{33}, (b) E_{11}, (c) ν_{13}, (d) G_{13}, (e) G_{12}.

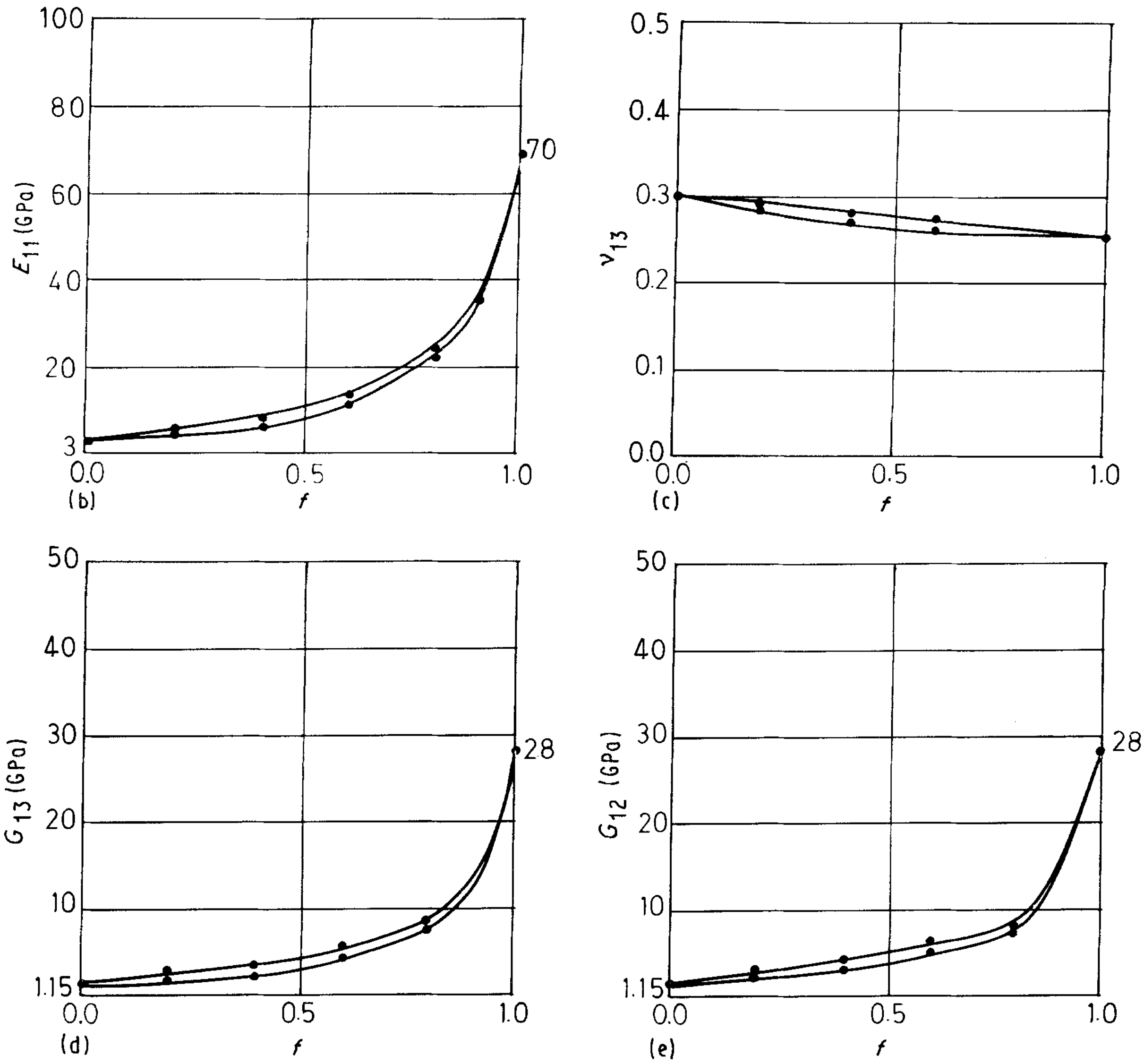

Fig. 4 Continued

the more sophisticated bounding methods. Fig. 4 presents the dependence of bounds of particular constants on the reinforcement factor *f*. All values were produced using the computer program ANREBO (ANisotropic REinforcement BOunds), which also allows the prediction of the elastic constants bounds for an anisotropic reinforcement.

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Received 6 July and accepted 11 August 1992